The Electronic Journal of Mathematics and Technology, Volume 9, Number 3, ISSN 1933-2823

Teaching numerical methods using CAS

Ildikó Perjési-Hámori
e-mail: perjesi@pmmik.pte.hu
University of Pécs
Pollack Mihály Faculty of Engineering and Information Technology
Pécs, Hungary

University of Pécs launched different MSc and BSc programs in engineering in 2013. The curriculum involves teaching Numerical methods. Our paper summarizes our experience obtained while teaching this subject. The focus of the subject was solving model problems using Maple, a Computer Algebra System (CAS), which sometimes substitutes the exact mathematical proofing. While solving model problems we tried to take advantage of the opportunities offered by the Maple computer algebra system. While composing the topics of the course, rapid development of computer algebra systems was taken into consideration. On the other hand, in this way students with limited mathematical skills are also able to understand more complex tasks, such as solutions of multivariate interpolations and regressions, or those of partial differential equations. In our paper we present some real-life examples from the course material and present some sample test questions using Maple T. A. test and assessment system.

1. Introduction

The University of Pécs, Pollack Mihály Faculty of Engineering and Information Technology offers special numerical mathematics courses for students of the MSc programs of information technology engineering, urban systems engineering and structural engineering in Hungarian and for the BSc and Erasmus students in English. In our courses we try to take into consideration the recommendations of SEFI's (European Society for Engineering Education) Mathematics Working Group [3].

Mathematical competence is the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role [5]. The recommendation for engineering students is to reach the following competences: thinking and reasoning mathematically, posing and solving mathematical problems, modelling mathematically, representing mathematical entities, handling mathematical symbols and formalism, communicating in, with, and about mathematics, making use of aids and tools. The required mathematical knowledge is divided into different levels.

The numerical methods play important role in the engineering practice. So these methods are not included separately in one level only, but they are built into each level. The chapters where the numerical methods are introduced are as follows:

- Level 0: Error in measurement
- Level 1: Least square curve fitting, Taylor series, numerical integration.
- Level 2: Numerical methods for first order ODE, Fourier series.
- Level 3: Numerical methods for ODE, Fourier analysis, finite element method

In our courses we summarize those topics which were mentioned during the basic mathematics courses and introduce some new methods. During these courses all problems and the subject materials are written and solved by using computer algebra. For homework, Maple T. A. test and assessment system was used.

In this paper firstly some mathematical-didactical aspects of CAS in higher mathematics are summarized, its different roles (as a computer program, as a didactical tool, as a mathematical assistant) are presented. In the next part the didactical questions of teaching numerical methods are discussed, why numerical methods can be used as a synthesis of the previously thought mathematical topics, and how these themes can be taught to students with different preliminary knowledge. Finally we summarize our experience and our plans for the future.

2. Planning a course using CAS

At the Department of Mathematics, University of Pécs we have used MAPLE (a CAS system) for teaching mathematics for 15 years. The MAPLE software and our worksheets are available for students via Intranet. The worksheets of the course are organized into one system by means of the "Table of Contents" MAPLE worksheet. In this page there are hyperlinks to the exercises and the exercises end with a hyperlink to the main sheet. So there is a close connection between the different parts of the course. The mathematical-didactical aspects, which were taken into account while planning this system, were as follows:

- The above system has to cover the syllabus of the course in the subject matter.
- The user must have the opportunity to become familiar with the special language of MAPLE. We allowed only the most necessary commands to be used.
- The new commands have to be distributed smoothly on the worksheets.
- Every worksheet has to contain textual explanations, but we have to avoid the extreme comments.
- According to the principle of spirality [12], first we introduced every new command through an easy mathematical problem.
- We wrote the programs taking into account the principle of gradation. We detail each steps of problem solving in the exercises that we could solve without the computer algebra system. Procedures were written from frequently used commands after the detailed explanations only. (White box – black box effect [1]).
- In the commands that are used only for visualization or rather where the algorithm of the solution passes the frame of the course, we used the black box – white box effect. It means there is an opportunity to understand each step of the solution for those students who want to do so. But everybody knows what the scheme of the problem solving is and what the result of the procedures is (principle of the operationalization).
- We have to avoid using CAS only for its own sake; it is only inferior to the mathematical subject matter.
- We have to take notice that the new technique demands students to become acquainted with new concepts.
- We have to profit from the graphical opportunity; to help the concepts meaning not only at the symbolical level but at the visualization level as well. Flexible visualization is the most important advantage of the CAS system.
- Exercises which were very difficult to solve without CAS become routine problems.
- We have to show symbolical calculations, first of all to adopt them to prove the identities.
- Due to the principle of consciousness we have to pay attention to the fact that mathematical meaning has priority over technical details. Our goal is that the user can adopt the concepts and procedures in new situations.

3. CAS – in reproduction, connection, reflection relationship

In our research group at the University of Pécs a lot of didactical projects are dealing with the role of computer algebra in teaching mathematics. We reflected on how the systemic use of technology may impact recruitment, entrance and retention [4], how we can connect the distance learning methods and CAS [6], [7], how can we use internet in mathematics education [8] and how it is used for different topics of mathematics [9], [10]. Furthermore, we organized international conferences on this topic [13], [14]. In the engineering society there are two different opinions about mathematics. One of them (mostly coming from architects) is that mathematics is only for mathematicians, engineers use computers and software to solve their problems. The other opinion is that engineers can calculate anything with the help of a simple calculator. Reflecting to these

opinions we think that computer algebra systems connect mathematicians and engineers in problem solving.
In Figure 1 some aspects of the importance of CAS are demonstrated.

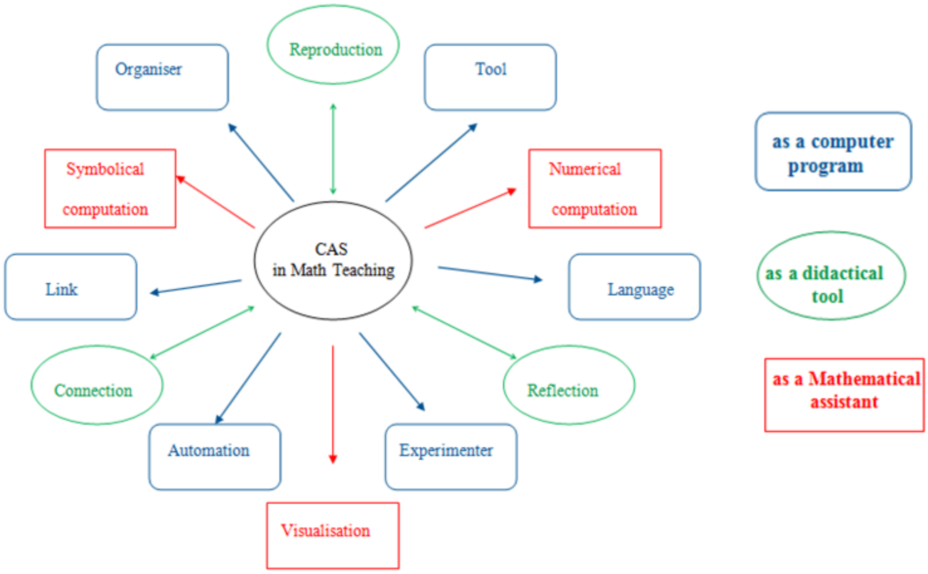


Figure 1: *The didactical aspects of CAS*

CAS as a *mathematical assistant*:
Applications of CAS increase the capability of algebraic, graphical and numerical representations of mathematical objects. Accordingly, by a variety of demonstrations of mathematical concepts the pragmatic and epistemological values [11] may also increase. To be able to precisely understand mathematical concepts we must use various representations or series of representations. CAS offers an opportunity for users to prepare this variety of representations, providing a flexible instrument for generating a multifaceted understanding of the concepts. For instance, experimental and discovery thinking can be assisted by well-designed and quickly generated procedures, models, and algorithms.

CAS as a *didactical tool*:
Both the field of mathematics and the mathematics curricula are structured into modules, complex and interconnected elements of knowledge that can be recalled from memory without consciously being aware of their internal structure. The fundamental problem of using CAS in teaching and learning is how to effectively develop cognitive schemes and a knowledge-representation net in particular modules. CAS-modularization presents an essentially new mode for knowledge development and heuristics. CAS environments enable reduction of complexity and the ease of computations; thus, enhance the connection of concepts and effective transfer of knowledge. From the block of representation net – depending on the actual problem – students can recall, repeat or reinforce the mathematics schemes.

CAS as a *computer program*:
A computer program has its own computer language. It sometimes causes difficulties to students, because they have to translate the problem to be solved not only from the everyday language into the language of mathematics, but from the everyday language into the program language and the results of the program into the language of mathematics as well.

If we have time at the beginning of a course which uses CAS, elementary introduction into the program language will help the user to organize their problem solving process, to make experiments, to automate some parts of the problem solving with the help of procedures and use CAS as an easy-to-use tool.

4. Didactical questions – numerical methods as a synthesis

In this paragraph applications from the topic of numerical mathematics are presented, which connect different themes of mathematics or connect real engineering problem and mathematics.

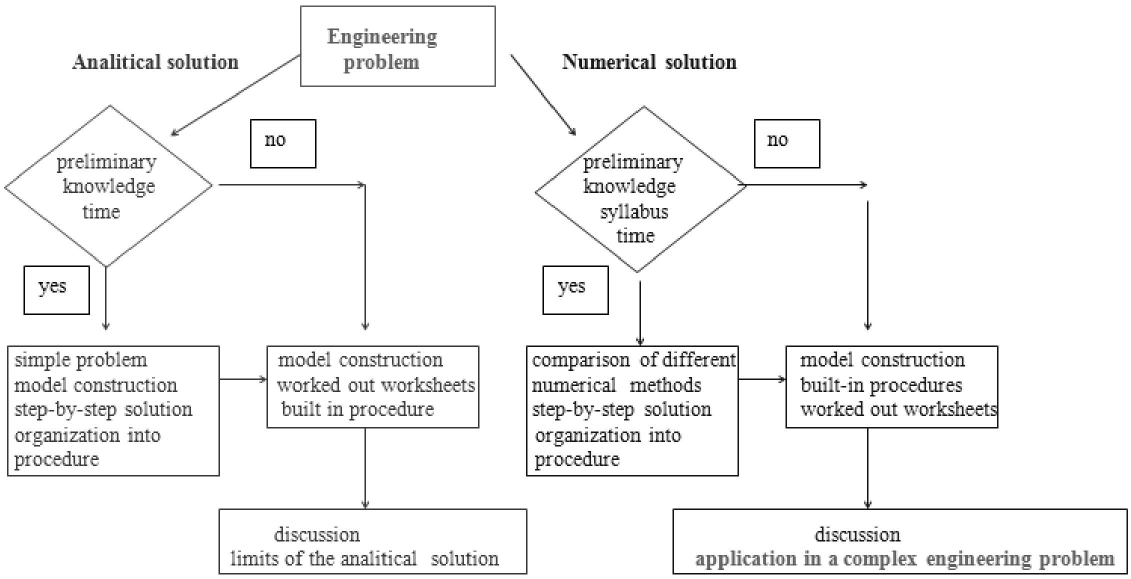


Figure 2: *Numerical methods in engineering problem solving.*

Figure 2 shows the algorithm of the way of engineering problem solving in a course curriculum. The roots depend on the type of engineering problem (it is solvable analytically or not), the preliminary knowledge of the students (step-by step solution or well-prepared worksheets). Finally the more complex application is required, taking advantage of the possibilities of CAS.

In Figure 3 the outline and the didactical tools of a chapter of the Numerical methods lecture notes are shown. This chapter presents the interaction between the deeper mathematical understanding and the possibilities of CAS. In this figure it is shown how the didactical aspect was used in teaching the least-square approximation. The role of the didactical tool is the most important part of this chapter (in the figure the colour is green). It is not because of using CAS, but of the property of numerical mathematics. The Mathematical assistant role (red) is a very significant part; the visualization, the symbolical and numerical computation possibilities transpose the main emphasis from the boring calculations into the meaning of the theory. As a computer program (marked with blue colour) it means that it is necessary to explain the new Maple commands, and as a tool the user can automate some parts of the procedure or Maple can automatize this process, using built-in tools.

After applying the mathematical concepts of extreme values of two variable functions and the solution of linear equation system it is possible to visualize the best line and the differences and calculate the sum of the difference squares, which leads to a deeper understanding of the concept of the least square method (Figure 4). In paragraph 8.3 we show the limit of the computer algebra systems. Figure 5 shows that the best fitting hyperbola leads to big errors in the approximation. If the approximate function is an $y = ax^b$ type, the *Maple LeastSquare* command cannot handle this approximation. The step-by-step solution gives a non-solvable non- linear system, so we have to use mathematical consideration, and we have to linearize the data structure.

After the introduction of the least square method in one variable using the well-known matrix algebra concepts, students have the possibility to generalize the concept into two variable problems. Without CAS, the generalized problem and the matrix operations were difficult and confusing. Using CAS we are able to concentrate on the mathematical theory.

In the next part of this chapter a few real-life problems are mentioned, which bring students closer to understand the importance of this theory.

▼ **8. Least- squares approximation**

▶ **8. 1. Coefficients of the best approximating line** *Connection: Mathematical concepts - Reproduction: Calculus of two-variable function,solution of linear system, Numerical computation, Language: new Maple commands*

▶ **8. 2. Least-square fitting by polynomial of higher degree**

▶ **8. 3. Linearization** *Visualisation, Experiment with curves, the limit of CAS, Connection: solving non-linear system, Reflection: to solve the problem by reversing a previously known problem*

▼ **8. 4. Coefficients usig linear algebra** *Connection: Mathematical concepts, Reproduction: Elements of linear algebra, Operations with matrixes, Symbolical computation*

▶ **8.4.1. Regression in one variable** *Reflection: coefficients using matrix and its transpose in a different meanings*

▶ **8.4.2. Multivariate regression** *Reflection: LinearAlgebra in new contact*

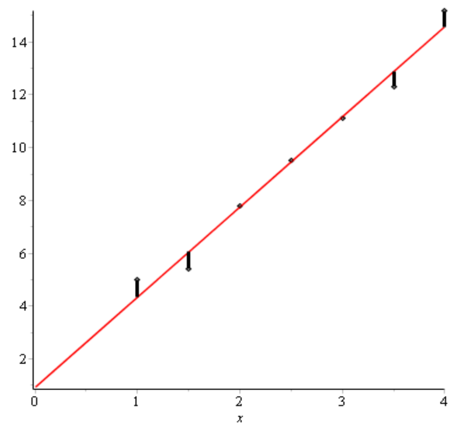
▶ **8. 5. Applications** *Link, mechanisator, tool*

▶ **8. 6. Built in Maple possibilities** *Reproduction, Tool, Mechanisator, Numerical computation*

▶ **8. 7. Fitting using arbitrary functions** *Connection: solving non-linear system, Visualisation,*

▶ **8. 8. Exercises**

Figure 3: *Didactical considerations in one chapter of the course.*



Determine the sum of the least-squares error and the deviations:

$$> \sum_{i=1}^{N} \left(subs(x = 'X'_i, y) - 'Y'_i\right)^2 = \sum_{i=1}^{N} \left(subs(x = X_i, y) - Y_i\right)^2$$

Figure 4: *Visualization of the least square method.*

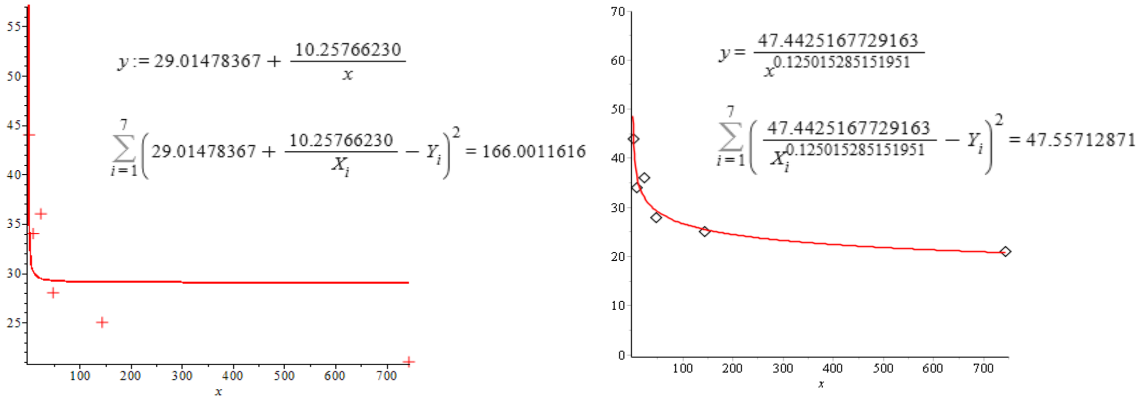


Figure 5: *The best fitting by hyperbola and power function*

5. Didactical questions – the set-up of a contact lesson using different levels and different students

The Numerical methods lecture notes are used in three different courses. In the next example we present the way of the teaching-learning process. Our students were MSc students, Erasmus students, and interested BSc students, who had strong motivation not only to pass this course but use it in their engineering practice. We used the results of the didactic of the distance learning, from where it is known, that self-regulated learning is an active, constructive process in which the student sets a goal of self-monitor, regulates and controls the learning process. Learning is an active, cognitive, constructive process. The three phases are the willing phase (motivation), the implementation phase and the feedback phase [2].

The examples are from the chapter of numerical methods of ordinary differential equations.

Example 1. For an elective course to graduate and Erasmus students (of different culture and mathematical background language problems): first we explained the mathematical concept, then solved easy exercise on the blackboard, after which a carefully prepared worksheet was introduced, and finally students solved easy exercises using Maple (only a few Maple command). For the numerical computation, first the step by step solution was used, organizing repetition statement only at the end of the contact lesson:

Problem: Give the analytical solution of the first order differential equation using Picard method.

$$y1(x) = y0 + \int_{x0}^{x} f(x, y0)\,\mathrm{d}x : \; y2(x) = y0 + \int_{x0}^{x} f(x, y1(x))\,\mathrm{d}x : \quad y3(x) = y0 + \int_{x0}^{x} f(x, y2(x))\,\mathrm{d}x :$$

We get a function sequence, and the limit of it (if it exists) is the particular solution of the differential equation. For a particular ODE:

```
de := diff(y(x), x) = x^2 + y(x); x0 := 0; y0 := 2;  partsol := dsolve([de, y(x0) = y0], y(x))
for i to 5 do
    f[i] := subs(y(x) = y[i - 1], rhs(de));
    y[i] := y0 + int(f[i], x = x0 ..x);
    fig[i] := plot([y[i], rhs(partsol)], x = -4 ..3, color = [blue,
   red]) :
end do:

figure := Matrix(2, 2, [seq(fig[i], i = 2 ..5)]) : plots[display](figure)
```

In Figure 6 some steps of the solutions are shown; on a real Maple worksheet the animation is very impressive.

Example 2. For an applied mathematics course to students of an MSc in urban systems programme (after the completion of various BSc programmes, only little mathematical knowledge): Presentation: Schematic mathematical concept, easy exercise on the blackboard, built-in tutorial, Maple DETools package, engineering applications.

Problem: Plan the shape of a channel, if we know the average speed of the water downfall of the surface, and the material constants of the channel. The differential equation of the problem from the civil engineering literature is

$$\frac{d}{dx}y(t) = \sqrt{0.66\,y(t)^2 - 1} \qquad y(0) := -1.5$$

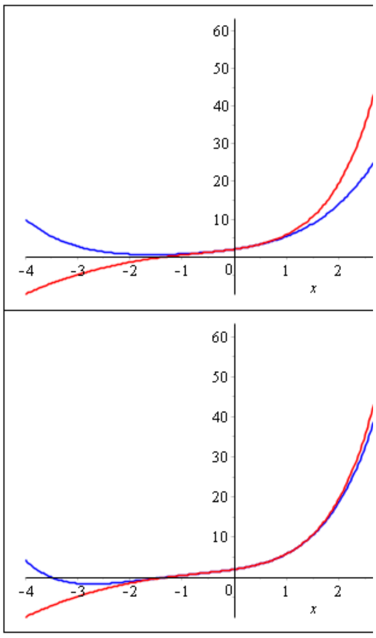


Figure 6: *Exact solution and 4 approximate solutions using Picard methods*

Maple has the possibilities to solve this ODE exactly, but the function is more complicated to handle it easily:

$$y(t) = \frac{1}{66}\,\frac{\sqrt{33}\left(-100 - 197\left(e^{\frac{1}{10}\sqrt{33}\,\sqrt{2}\,t}\right)^2 + 3\left(e^{\frac{1}{10}\sqrt{33}\,\sqrt{2}\,t}\right)^2\sqrt{33}\,\sqrt{97}\right)}{e^{\frac{1}{10}\sqrt{33}\,\sqrt{2}\,t}\left(3\sqrt{33} - \sqrt{97}\right)}$$

With the Maple *Tools* menu *ODE analyser* command students are able to solve the equations numerically or symbolically. Didactically it is a black box, but sometimes it can be very useful in the engineering practice (Figure 7.).

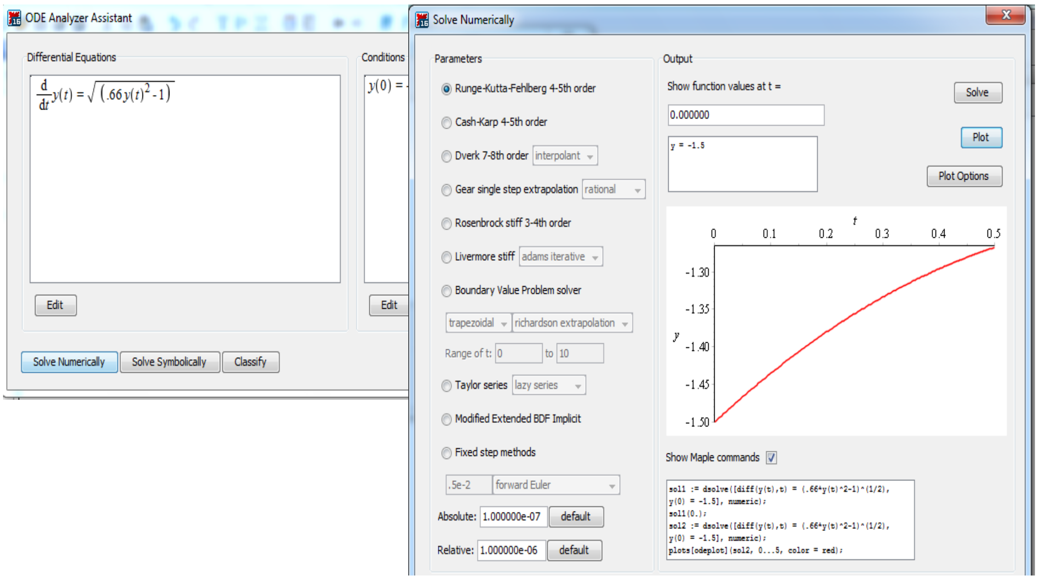


Figure 7: *Maple ODE analyser*

Example 3. For MSc students of information technology engineering (with a well-defined mathematical and programming background):
Presentation: Mathematical modelling, easy exercise on the blackboard, step-by-step solution, organize procedures, experiments with the error level, well-prepared worksheet only for homework.

Problem: the movement of a pendulum, when the initial deflection is 90^o. The differential equation of the movement is

$$\frac{d^2}{dt^2}\varphi(t) = -10\cdot\sin(\varphi(t)) \qquad \varphi(0) = \frac{\pi}{2}, D(\varphi)(0) = 0$$

where φ is the deflection angle, t is the time. After introducing step-by-step numerical solutions, the built-in procedure is used for solving the problem:

$$partsol := dsolve\left(\left\{de, \varphi(0) = \frac{\pi}{2}, D(\varphi)(0) = 0\right\}, \varphi(t), numeric\right)$$
$$T := [seq(rhs(partsol(0.01\cdot i)[1]), i = 0..500)]:$$
$$\phi := [seq(rhs(partsol(0.01\cdot i)[2]), i = 0..500)]:$$

Students are able to write a short procedure to visualize the movement of the pendulum (Figure 8).

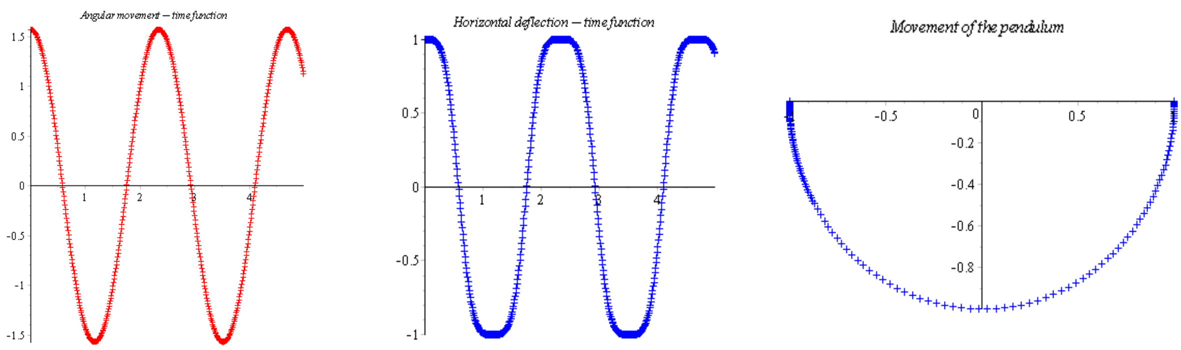


Figure 8: *Movement of a pendulum*

6. Next step – Maple T. A. question bank

In the higher education system *in the past* courses were divided into lectures and seminars, students had several contact lessons, seminars were held for small groups, the exercises were solved using chalk and pencil on the blackboard and in exercise books, students made hand written notes, there were oral exams and a lot of face to face contact between students and instructors.

At present, students have less contact lessons, there are bigger groups, we use handwritten, printed and on-line notes, computer algebra systems, there are written exams, and very little personal contact.

In the future, self-regulated learning, web-based learning, (semi) personal contact, and web-based learning and exams will have a greater importance.

One possibility: Maple T. A. web-based test and assessment system, powered by Maple, where students can assess their own knowledge of a particular topic, and teachers can get feedback from the students' level of knowledge.

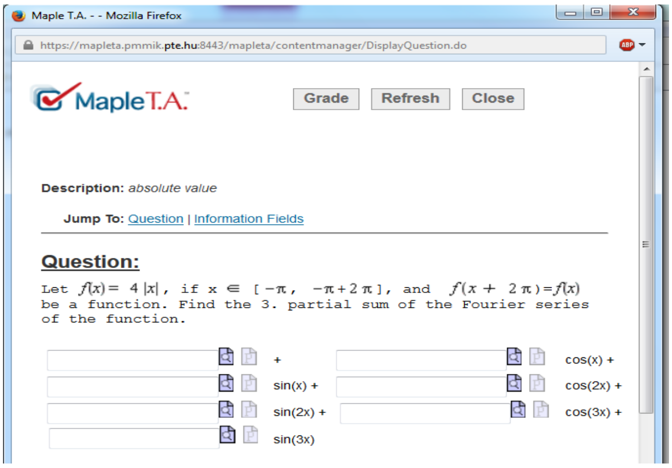


Figure 9: *A question from the question bank*

Levels of application:

- simple questions, little Maple and programming knowledge
- complex questions, middleware Maple and little programming knowledge
- difficult Maple packages, complicated response programming

At the University of Pécs we developed a question bank for practicing numerical methods. In Figure 9 there is a sample test question in the theme of the Fourier series.

7. Conclusions

Our experience is that we have to find the place of Numerical course in the engineering curriculum and we have to use the methods of blended learning and the didactics of CAS. Using well-prepared lecture notes, we have to take into consideration the students' different background knowledge in mathematics, in programing and in English: each course has its own way.

We have to synthesize and apply the basic mathematical concepts in a different way, to rebuild the knowledge representation net, to make students to generalize mathematical concepts, to connect various topics of mathematics. Students have to become closer to know how to apply the knowledge in everyday life and in engineering.

From the students' point of view, it is important that the mathematical concepts and procedures which are needed for the engineering tasks but are not taught due to lack of the time could be introduced into the syllabus. CAS increases the significance of heuristic problem-solving strategies. Its use in solving exercises makes them become routine tasks for students who found them very difficult to solve earlier. It inspires students to get to know new mathematical concepts. Its effective use requires new approaches in mathematics didactics.

Considering the possibilities of the new strategies, test results will be better and students' bad preconceptions about mathematics will decrease. Prerequisite of the strategy is the students' basic computational knowledge. These new type of teaching-learning process helps the integration of mathematics and special engineering subjects.

8. References

[1] B. Buchberger, *Should Students Learn Integration Rules*?. ACM SIGSAM Bulletin 24(1), pp. 10-17. January 1990.

[2] DL Butler, PH Winne, *Feedback and self-regulated learning: A theoretical synthesis*-Review of educational research, 65 (3), 245-281 1995

[3] http://www.sefi.be/wp-content/uploads/Competency%20based%20curriculum%20incl%20ads.pdf

[4] E. Muller, C. Buteau , M. Klincsik, I. Perjési-Hámori, Cs. Sárvári, *Systemic integration of evolving technologies in undergraduate mathematics education and its impact on student retention*, International Journal of Mathematical Education in Science and Technology iJMEST Vol. 40, No. 1, 15 January 2009, 139-155

[5] M. Niss, *Quantitative literacy and mathematical competencies*. In B.L. Madison , L.A. Steen (Eds.), Quantitative literacy : why numeracy matters for schools and colleges. Princeton : National Council on Education and the Disciplines,215-220, 2003.

[6] I. Perjési-Hámori, M. Klincsik, *An Interactive Method of Teaching Vectorcalculus via Internet* ODL Networking for Quality Learning, Lisbon, Conference Proceedings 67-71. 2000.

[7] I. Perjési-Hámori, *Application of CAS for teaching integral-transforming theorems* Zentralblatt für Didaktik der Mathematik 35 (2) 43-47 2003.

[8] I Perjési-Hámori, Cs. Sárvári, M. Klincsik, *Using CAS and Internet In Teaching Mathematics At University of Pécs (Hungary*) EIESTA'03, International Conference on education and Information systems: Technologies and Applications, Orlando, Conference Proceedings 17-23. 2003.

[9] I. Perjési-Hámori, *Experiences using CAS and multimedia in teaching Vectorcalculus* Teaching Mathematics and Computer Science 1/2 , 363-382. 2003.

[10] I. Perjési-Hámori, Cs. Sárvári, *Teaching Fourier series, partial differential equations and their applications with help of computer algebra system* Teaching Mathematics and Computer Science 7/1 , 51-68, 2009.

[11] C. Sárvári, *Pragmatic, epistemological, and heuristic values in CAS enhanced mathematics education* In M. Kourkoulos, G. Troulis, C. Tzanakis, Proceedings of the 4th Colloquium on the Didactics of Mathematics, University of Crete, pp. 217-226, 2005.

[12] J. Bruner, *The Process of Education*. Cambridge, MA: Harvard University Press, 1960

[13] http://matserv.pmmf.hu/cadgme/

[14] http://matep.pmmik.pte.hu/en/